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# Some geometric properties of the metric space $V[\lambda, p]$

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Full list of author information is available at the end of the article**Abstract**

In this study, we consider the space  $V[\lambda, p]$  with an invariant metric. Then, we examine some geometric properties of the linear metric space  $V[\lambda, p]$  such as property  $(\beta)$ , property  $(H)$  and  $k$ -NUC property.

**MSC:** 40A05; 46A45; 46B20**Keywords:** linear metric space; Luxemburg norm; de la Vallée-Poussin mean;  $k$ -NUC property; property  $(H)$ 

## 1 Introduction

Let  $X$  be a vector space over the scalar field of real numbers and  $d$  be an invariant metric on  $X$ . We denote  $B_d(X)$  and  $S_d(X)$  as follows:

$$B_d(X) = \{x \in X : d(x, 0) \leq r\} \quad \text{and}$$

$$S_d(X) = \{x \in X : d(x, 0) = r\}.$$

Let  $(X, d)$  be a linear metric space and  $B_d(X)$  (resp.,  $S_d(X)$ ) be a closed unit ball (resp., the unit sphere) of  $X$ . A linear metric space  $(X, d)$  has property  $(\beta)$  if and only if for each  $r > 0$  and  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for each element  $x \in B_d(0, r)$  and each sequence  $(x_n)$  in  $B_d(0, r)$  with  $\text{sep}(x_n) \geq \varepsilon$ , there is an index  $k$  for which  $d(\frac{x+x_k}{2}, 0) \leq 1 - \delta$ , where  $\text{sep}(x_n) = \inf\{d(x_n, x_m) : n \neq m\} > \varepsilon$  [1]. If for each  $x \in S_d(0, r)$  and  $(x_n) \subset S_d(0, r)$ ,  $x_n \xrightarrow{w} x$  implies  $x_n \rightarrow x$ , a linear metric space  $(X, d)$  is said to have property  $(H)$ . Let  $k \geq 2$  be an integer. A linear metric space  $(X, d)$  is said to be  $k$ -nearly uniform convex ( $k$ -NUC) if for every  $\varepsilon > 0$  and  $r > 0$ , there exists  $\delta > 0$  such that for any sequence  $(x_n) \subset B_d(0, r)$  with  $\text{sep}(x_n) \geq \varepsilon$ , there are  $s_1, s_2, \dots, s_k$  such that  $d(\frac{x_{s_1} + x_{s_2} + \dots + x_{s_k}}{k}, 0) \leq r - \delta$  [2]. These properties have been studied by Mongkolkeha and Pumam [3], Sanhan and Suantai [4], Cui *et al.* [5] and Cui and Hudzik [6].

Ahuja *et al.* [7] introduced the notions of strict convexity and U.C.I (uniform convexity) in linear metric spaces which are generalizations of the corresponding concepts in linear normed spaces. Later, Sastry and Naidu [8] introduced the notions of U.C.II and U.C.III in linear metric spaces and showed that these three forms are not always equivalent. Further, Junde *et al.* [9, 10] showed that if a linear metric space is complete and U.C.I, then it is reflexive.

In summability theory, de la Vallée-Poussin mean was first used to define the  $(V, \lambda)$ -summability by Leindler [11].  $(V, \lambda)$ -summable sequences have been studied by many au-

thors including Et *et al.* [12, 13], Savas [14–18], Savas and Malkowsky [19] and Şimsek *et al.* [20, 21]. Let  $\Lambda = (\lambda_k)$  be a nondecreasing sequence of positive real numbers tending to infinity and let  $\lambda_1 = 1$  and  $\lambda_{k+1} \leq \lambda_k + 1$ . The generalized de la Vallée-Poussin mean is defined by  $t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k$ , where  $I_n = [n - \lambda_n + 1, n]$  for  $n = 1, 2, \dots$ . A sequence  $x = (x_k)$  is said to be  $(V, \lambda)$ -summable to a number  $\ell$  if  $t_n(x) \rightarrow \ell$  as  $n \rightarrow \infty$ . If  $\lambda_n = n$ , then  $(V, \lambda)$ -summability is reduced to Cesàro summability.

Let  $w$  be the space of all real sequences. Let  $p = (p_k)$  be a bounded sequence of positive real numbers. Şimşek *et al.* [20] defined the space  $V[\lambda, p]$  as follows:

$$V[\lambda, p] = \left\{ x = (x_k) \in \omega : \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_j| \right)^{p_k} < \infty \right\}.$$

If  $\lambda_k = k$ , then  $V[\lambda, p] = \text{ces}(p)$  [22]. If  $\lambda_k = k$  and  $p_k = p$  for all  $k \in \mathbb{N}$ , then  $V[\lambda, p] = \text{ces}_p$  [23]. Paranorm on  $V[\lambda, p]$  is given by

$$h(x) = \left( \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_j| \right)^{p_k} \right)^{\frac{1}{M}},$$

where  $M = \max\{1, H\}$  and  $H = \sup p_k$ . If  $p_k = p$  for all  $k \in \mathbb{N}$ , the notation  $V_p(\lambda)$  is used in place of  $V[\lambda, p]$  and the norm on  $V_p(\lambda)$  is as follows:

$$\|x\|_{V_p(\lambda)} = \left( \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_j| \right)^p \right)^{\frac{1}{p}}.$$

$\rho : V_p[\lambda, p] \rightarrow [0, \infty]$ ,  $\rho(x) = (\sum_{k=1}^{\infty} (\frac{1}{\lambda_k} \sum_{j \in I_k} |x_j|)^{p_k})$  is a modular on  $V_p[\lambda, p]$  and the Luxemburg norm on  $V_p[\lambda, p]$  is defined by  $\|x\|_L = \inf\{\sigma > 0 : \rho(\frac{x}{\sigma}) \leq 1\}$  for all  $x \in V_p[\lambda, p]$ . The Amemiya norm on the space  $V_p[\lambda, p]$  can be similarly introduced as follows:

$$\|x\|_A = \inf_{\sigma > 0} \frac{1}{\sigma} (1 + \rho(\sigma x)) \quad \text{for all } x \in V_p[\lambda, p].$$

## 2 Main results

In this part of the paper, our main purpose is to define a metric on  $V[\lambda, p]$  and show that  $V[\lambda, p]$  possesses property  $(\beta)$ , property  $(H)$  and  $k$ -NUC property. Let  $p = (p_k)$  be a bounded sequence of real numbers with  $p_k > 1$  for all  $k \in \mathbb{N}$ . The mapping  $d(x, y) = (\sum_{k=1}^{\infty} (\frac{1}{\lambda_k} \sum_{j \in I_k} |x(j) - y(j)|)^{p_k})^{1/H}$  is a metric on the space  $V[\lambda, p]$ , where  $M = \max(1, H = \sup p_k)$  and  $m = \inf p_k$  since the function  $|t|^p$  is convex for  $p > 1$ . First, we will show that the space  $V[\lambda, p]$  has property  $(\beta)$  under the above metric. To do this, we need the following two lemmas. To prove these lemmas, we use the technique given in Sanhan and Mongkolkeha [1].

**Lemma 2.1** *Let  $y, z \in (V[\lambda, p], d)$ . If  $\beta \in (0, 1)$ , then*

$$(d(y + z, 0))^M \leq (d(y, 0))^M + 2^M \beta (d(y, 0))^M + \frac{2^M}{\beta^{M-1}} (d(z, 0))^M.$$

*Proof* Let  $y, z \in (V[\lambda, p], d)$  and  $0 < \beta < 1$ . Then

$$\begin{aligned}
 (d(y + z, \mathbf{0}))^M &= \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |y(j) + z(j)| \right)^{p_k} \\
 &\leq \sum_{k=1}^{\infty} \left( (1 - \beta) \frac{1}{\lambda_k} \sum_{j \in I_k} |y(j)| + \beta \frac{1}{\lambda_k} \sum_{j \in I_k} \left| y(j) + \frac{z(j)}{\beta} \right| \right)^{p_k} \\
 &\leq (1 - \beta) \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |y(j)| \right)^{p_k} + \beta \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} \left| y(j) + \frac{z(j)}{\beta} \right| \right)^{p_k} \\
 &\leq \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |y(j)| \right)^{p_k} + 2^M \beta \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |y(j)| \right)^{p_k} \\
 &\quad + 2^M \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} \left| \frac{z(j)}{\beta} \right| \right)^{p_k} \\
 &\leq \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |y(j)| \right)^{p_k} + 2^M \beta \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |y(j)| \right)^{p_k} \\
 &\quad + \frac{2^M}{\beta^{M-1}} \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |z(j)| \right)^{p_k} \\
 &= (d(y, \mathbf{0}))^M + 2^M \beta (d(y, \mathbf{0}))^M + \frac{2^M}{\beta^{M-1}} (d(z, \mathbf{0}))^M. \quad \square
 \end{aligned}$$

**Lemma 2.2** Let  $y, z \in (V[\lambda, p], d)$ . Then for any  $\varepsilon > 0$  and  $L > 0$ , there exists  $\delta > 0$  such that

$$|(d(y + z, \mathbf{0}))^M - (d(y, \mathbf{0}))^M| < \varepsilon,$$

where  $(d(y, \mathbf{0}))^M \leq L$  and  $(d(z, \mathbf{0}))^M \leq \delta$ .

*Proof* Let  $\varepsilon > 0$  and  $L > 0$ . For  $\beta = \frac{\varepsilon}{2^{M+1}(L+\varepsilon)}$ , we take  $\delta = \frac{\varepsilon \beta^{M-1}}{2^{M+1}}$ . From Lemma 2.1, we have

$$\begin{aligned}
 (d(y + z, \mathbf{0}))^M &\leq (d(y, \mathbf{0}))^M + 2^M \beta (d(y, \mathbf{0}))^M + \frac{2^M}{\beta^{M-1}} (d(z, \mathbf{0}))^M \\
 &\leq (d(y, \mathbf{0}))^M + 2^M \beta L + \frac{2^M}{\beta^{M-1}} \delta \\
 &\leq (d(y, \mathbf{0}))^M + 2^M \frac{\varepsilon}{2^{M+1}} \frac{L}{L + \varepsilon} + \frac{2^M}{\beta^{M-1}} \frac{\varepsilon \beta^{M-1}}{2^{M+1}} \\
 &\leq (d(y, \mathbf{0}))^M + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
 &\leq (d(y, \mathbf{0}))^M + \varepsilon \tag{2.1}
 \end{aligned}$$

and

$$\begin{aligned}
 (d(y, \mathbf{0}))^M &\leq (d(y + z, \mathbf{0}))^M + 2^M \beta (d(y + z, \mathbf{0}))^M + \frac{2^M}{\beta^{M-1}} (d(-z, \mathbf{0}))^M \\
 &\leq (d(y + z, \mathbf{0}))^M + 2^M \beta ((d(y, \mathbf{0}))^M + \varepsilon) + \frac{2^M}{\beta^{M-1}} \delta
 \end{aligned}$$

$$\begin{aligned}
 &\leq (d(y+z, \mathbf{0}))^M + 2^M \beta(L+\varepsilon) + \frac{2^M}{\beta^{M-1}} \frac{\varepsilon \beta^{M-1}}{2^{M+1}} \\
 &= (d(y+z, \mathbf{0}))^M + 2^M \frac{\varepsilon}{2^{M+1}(L+\varepsilon)} (L+\varepsilon) + \frac{\varepsilon}{2} \\
 &= (d(y+z, \mathbf{0}))^M + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
 &= (d(y+z, \mathbf{0}))^M + \varepsilon.
 \end{aligned} \tag{2.2}$$

From (2.1) and (2.2), we obtain that  $|(d(y+z, \mathbf{0}))^M - (d(y, \mathbf{0}))^M| < \varepsilon$ .  $\square$

**Theorem 2.3** *The space  $(V[\lambda, p], d)$  has property  $(\beta)$ .*

*Proof* Let  $\varepsilon > 0$  and  $(x_n) \subset B(V[\lambda, p], d)$  such that  $\text{sep}(x_n) \geq \varepsilon$  and  $x \in B(V[\lambda, p], d)$ . We take  $y^N = (0, 0, \dots, 0, \sum_{k=1}^N y(k), y(N+1), y(N+2), \dots)$ . By using the diagonal method, we can find a subsequence  $(x_{n_r})$  of  $(x_n)$  for each  $N \in \mathbb{N}$  such that  $(x_{n_r}(k))$  converges for each  $k \in \mathbb{N}$  with  $1 \leq k \leq N$ , since  $(x_n(k))_{k=1}^\infty$  is bounded for each  $k \in \mathbb{N}$ . Therefore, there is  $t_N \in \mathbb{N}$  for each  $N \in \mathbb{N}$  such that  $\text{sep}((x_{n_r}^N)_{r \geq t_N}) \geq \varepsilon$ . So, there is a sequence of positive integers  $(t_N)_{N=1}^\infty$  with  $t_1 < t_2 < t_3 \dots$  such that  $d(x_{t_N}^N, \mathbf{0}) \geq \frac{\varepsilon}{2}$  for all  $N \in \mathbb{N}$ . Then there exists  $\kappa > 0$  such that for all  $N \in \mathbb{N}$ ,

$$\sum_{k=N}^\infty \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_{t_N}(j)| \right)^{p_k} \geq \kappa. \tag{2.3}$$

By Lemma 2.2, there exists  $\delta_0$  such that

$$|(d(y+z, \mathbf{0}))^M - (d(y, \mathbf{0}))^M| < \frac{\kappa}{2^m}, \tag{2.4}$$

where  $(d(y, \mathbf{0}))^M < j^M$  and  $(d(z, \mathbf{0}))^M \leq \delta_0$ . There exists  $N_1 \in \mathbb{N}$  such that  $(d(x^{N_1}, \mathbf{0}))^M \leq \delta_0$  if  $x \in B(V[\lambda, p])$  and  $(d(x, \mathbf{0}))^M \leq \delta_0$ . Let us take  $y = x_{t_{N_1}}^{N_1}$  and  $z = x^{N_1}$ . Hence, we have

$$\sum_{k=N_1}^\infty \left( \frac{1}{\lambda_k} \sum_{j \in I_k} \left| \frac{x(j) + x_{t_{N_1}}(j)}{2} \right| \right)^{p_k} \leq \sum_{k=N_1}^\infty \left( \frac{1}{\lambda_k} \sum_{j \in I_k} \left| \frac{x_{t_{N_1}}(j)}{2} \right| \right)^{p_k} + \frac{\kappa}{2^m}. \tag{2.5}$$

From (2.3), (2.4), (2.5) and by using the convexity of the function  $f(t) = |t|^{p_k}$  for all  $k \in \mathbb{N}$ , we obtain that

$$\begin{aligned}
 \left( d\left(\frac{y+z}{2}, \mathbf{0}\right) \right)^M &= \sum_{k=1}^\infty \left( \frac{1}{\lambda_k} \sum_{j \in I_k} \left| \frac{x(j) + x_{t_{N_1}}(j)}{2} \right| \right)^{p_k} \\
 &= \sum_{k=1}^{N_1-1} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} \left| \frac{x(j) + x_{t_{N_1}}(j)}{2} \right| \right)^{p_k} + \sum_{k=N_1}^\infty \left( \frac{1}{\lambda_k} \sum_{j \in I_k} \left| \frac{x(j) + x_{t_{N_1}}(j)}{2} \right| \right)^{p_k} \\
 &\leq \sum_{k=1}^{N_1-1} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} \left| \frac{x(j) + x_{t_{N_1}}(j)}{2} \right| \right)^{p_k} + \sum_{k=N_1}^\infty \left( \frac{1}{\lambda_k} \sum_{j \in I_k} \left| \frac{x_{t_{N_1}}(j)}{2} \right| \right)^{p_k} + \frac{\kappa}{2^m} \\
 &\leq \frac{1}{2} \sum_{k=1}^{N_1-1} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x(j)| \right)^{p_k} + \frac{1}{2} \sum_{k=1}^{N_1-1} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_{t_{N_1}}(j)| \right)^{p_k}
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2^m} \sum_{k=N_1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_{t_{N_1}}(j)| \right)^{p_k} + \frac{\kappa}{2^m} \\
 & \leq \frac{1}{2} \sum_{k=1}^{N_1-1} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x(j)| \right)^{p_k} + \frac{1}{2} \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_{t_{N_1}}(j)| \right)^{p_k} \\
 & \quad - \frac{2^m - 2}{2^{m+1}} \sum_{k=N_1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_{t_{N_1}}(j)| \right)^{p_k} + \frac{\kappa}{2^m} \\
 & < \frac{j^M}{2} + \frac{j^M}{2} - \frac{2^m - 2}{2^{m+1}} \kappa + \frac{\kappa}{2^m} \\
 & = j^M - \frac{\kappa}{2}.
 \end{aligned}$$

Therefore, we have  $d(\frac{y+z}{2}, \mathbf{0}) < (j^M - \frac{\kappa}{2})^{1/M} < j - \delta$  whenever  $\delta \in (0, j - (j^M - \frac{\kappa}{2})^{1/M})$ . Consequently, the space  $(V[\lambda, p], d)$  possesses property  $(\beta)$ .  $\square$

Now, we will show that the space  $(V[\lambda, p], d)$  has  $k$ -NUC property.

**Theorem 2.4** *The space  $V[\lambda, p]$  is  $k$ -NUC for any integer  $k \geq 2$ .*

*Proof* Let  $\varepsilon > 0$  and  $(x_n) \subset B_d(V[\lambda, p])$  with  $\text{sep}(x_n) \geq \varepsilon$ . For each  $m \in \mathbb{N}$ , let

$$x_n^m = (0, 0, \dots, x_n(m), x_n(m+1), \dots). \quad (2.6)$$

Since the sequence  $(x_n(i))_{i=1}^{\infty}$  is bounded for each  $i \in \mathbb{N}$ , by using the diagonal method, we can find a subsequence  $(x_{n_l})$  of  $(x_n)$  such that  $(x_{n_l}(k))$  converges for each  $k \in \mathbb{N}$ . Therefore, there is an increasing sequence  $t_m$  with  $\text{sep}((x_{n_l}^m)_{l > t_m}) \geq \varepsilon$ . Hence, there exists a sequence of positive integers  $(r_m)_{m=1}^{\infty}$  with  $r_1 < r_2 < r_3 < \dots$  such that  $d(x_{r_m}^m, \mathbf{0}) \geq \frac{\varepsilon}{2}$  for all  $m \in \mathbb{N}$ . Then there is  $\zeta > 0$  such that

$$\sum_{k=m}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_{r_m}^m| \right)^{p_k} \geq \zeta. \quad (2.7)$$

Let  $\alpha > 0$  such that  $1 < \alpha < \lim_{k \rightarrow \infty} \inf p_k$ . Let  $\varepsilon_1 = \frac{n^{\alpha-1}-1}{(n-1)n^{\alpha}} \frac{\zeta}{2}$  for  $k \geq 2$ . From Lemma 2.2, there is a  $\delta > 0$  such that

$$|(d(y+z, \mathbf{0}))^M - (d(y, \mathbf{0}))^M| < \varepsilon_1, \quad (2.8)$$

where  $(d(y, \mathbf{0}))^M < r^M$  and  $(d(z, \mathbf{0}))^M \leq \delta$ . Then there exist positive integers  $m_i$  ( $i = 1, 2, \dots, n-1$ ) with  $m_1 < m_2 < \dots < m_{n-1}$  such that  $d(x_i^{m_i}, \mathbf{0}) \leq \delta$ . Now, define  $m_n = m_{n-1} + 1$ . Then we have  $d(x_{r_{m_n}}^{m_n}, \mathbf{0}) \geq \zeta$  for all  $m \in \mathbb{N}$ . For  $1 \leq i \leq n-1$ , let  $s_i = i$  and  $s_n = r_{m_n}$ . By using (2.6), (2.7), (2.8) and the convexity of the function  $f_i(u) = |u|^{p_i}$  ( $i \in \mathbb{N}$ ), we obtain

$$\begin{aligned}
 & \left( d \left( \frac{x_{s_1} + x_{s_2} + \dots + x_{s_n}}{n}, \mathbf{0} \right) \right)^M \\
 & = \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} \left| \frac{x_{s_1}(j) + \dots + x_{s_n}(j)}{n} \right| \right)^{p_k}
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^{m_1} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} \left| \frac{x_{s_1}(j) + \cdots + x_{s_n}(j)}{n} \right| \right)^{p_k} + \sum_{k=m_1+1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} \left| \frac{x_{s_1}(j) + \cdots + x_{s_n}(j)}{n} \right| \right)^{p_k} \\
&\leq \sum_{k=1}^{m_1} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} \left| \frac{x_{s_1}(j) + \cdots + x_{s_n}(j)}{n} \right| \right)^{p_k} + \sum_{k=m_1+1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} \left| \frac{x_{s_1}(j) + \cdots + x_{s_n}(j)}{n} \right| \right)^{p_k} + \varepsilon_1 \\
&\leq \sum_{k=1}^{m_1} \frac{1}{n} \sum_{i=1}^n \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_{s_i}(j)| \right)^{p_k} + \sum_{k=m_1+1}^{m_2} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} \left| \frac{x_{s_2}(j) + x_{s_3}(j) + \cdots + x_{s_n}(j)}{n} \right| \right)^{p_k} \\
&\quad + \sum_{k=m_2+1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} \left| \frac{x_{s_3}(j) + x_{s_4}(j) + \cdots + x_{s_n}(j)}{n} \right| \right)^{p_k} + 2\varepsilon_1 \\
&\leq \sum_{k=1}^{m_1} \frac{1}{n} \sum_{i=1}^n \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_{s_i}(j)| \right)^{p_k} + \sum_{k=m_1+1}^{m_2} \frac{1}{n} \sum_{i=2}^n \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_{s_i}(j)| \right)^{p_k} \\
&\quad + \sum_{k=m_2+1}^{m_3} \frac{1}{n} \sum_{i=3}^n \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_{s_i}(j)| \right)^{p_k} + \cdots + \sum_{k=m_{n-1}+1}^{m_n} \frac{1}{n} \sum_{i=n-1}^n \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_{s_i}(j)| \right)^{p_k} \\
&\quad + \sum_{k=m_n+1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} \left| \frac{x_{s_n}(j)}{n} \right| \right)^{p_k} + (n-1)\varepsilon_1 \\
&\leq \left( \frac{(d(x_{s_1}, \theta))^M + (d(x_{s_2}, \theta))^M + \cdots + (d(x_{s_n}, \theta))^M}{n} \right) + \frac{1}{n} \sum_{k=1}^{m_n} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_{s_n}(j)| \right)^{p_k} \\
&\quad + \sum_{k=m_n+1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} \left| \frac{x_{s_n}(j)}{n} \right| \right)^{p_k} + (n-1)\varepsilon_1 \\
&\leq \frac{n-1}{n} r^M + \frac{1}{n} \sum_{k=1}^{m_n} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_{s_n}(j)| \right)^{p_k} + \frac{1}{n^\alpha} \sum_{k=m_n+1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} \left| \frac{x_{s_n}(j)}{n} \right| \right)^{p_k} + (n-1)\varepsilon_1 \\
&\leq r^M - \frac{r^M}{n} + \frac{1}{n} \left( r^M - \sum_{k=m_n+1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_{s_n}(j)| \right)^{p_k} \right) \\
&\quad + \frac{1}{n^\alpha} \sum_{k=m_n+1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} \left| \frac{x_{s_n}(j)}{n} \right| \right)^{p_k} + (n-1)\varepsilon_1 \\
&\leq r^M + (n-1)\varepsilon_1 - \left( \frac{n^{\alpha-1} - 1}{n^\alpha} \right) \zeta \\
&\leq r^M + (n-1) \frac{n^{\alpha-1} - 1}{n^\alpha(n-1)} \left( \frac{\zeta}{2} \right) - \left( \frac{n^{\alpha-1} - 1}{n^\alpha} \right) \zeta \\
&= r^M - \left( \frac{n^{\alpha-1} - 1}{n^\alpha} \right) \left( \frac{\zeta}{2} \right).
\end{aligned}$$

Thus, we have  $d(\frac{x_{s_1}(j)+x_{s_2}(j)+\cdots+x_{s_n}(j)}{n}, \mathbf{0}) < (r^M - (\frac{n^{\alpha-1}-1}{n^\alpha})\frac{\zeta}{2})^{1/M} < r - \delta$  for  $\delta \in (0, r - (r^M - (\frac{n^{\alpha-1}-1}{n^\alpha})\frac{\zeta}{2})^{1/M})$ . Hence,  $(V[\lambda, p], d)$  is  $k$ -NUC.  $\square$

Since  $k$ -NUC implies NUC and NUC implies property  $(H)$ , by using the previous theorem, we can give the following result.

**Corollary 2.5** *The space  $(V[\lambda, p], d)$  has property  $(H)$ .*

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

MC, MK and ME have contributed to all parts of the article. All authors read and approved the final manuscript.

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